ESC194 Unit 9

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1 Separable Equations

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \to \int h(y)dy = \int g(x)dx$$

Separable equation.

$$\frac{dy}{dx} = \pm ky \to g = Ce^{\pm kx}$$

2 Compound Interest

Annual

$$V(t) = V(0)(1+i)^t$$

t = final value V(0) = initial value i = interest rate as decimal

Semi-Annual

$$v(t) = V(0)(1+\frac{i}{2})^{2t}$$

n = 365, daily interest.

But consider the limit:

$$\lim_{x\to\infty}(1+\frac{i}{n})^{nt} = \lim_{x\to\infty}(1+\frac{1}{n/i})^{nt}$$

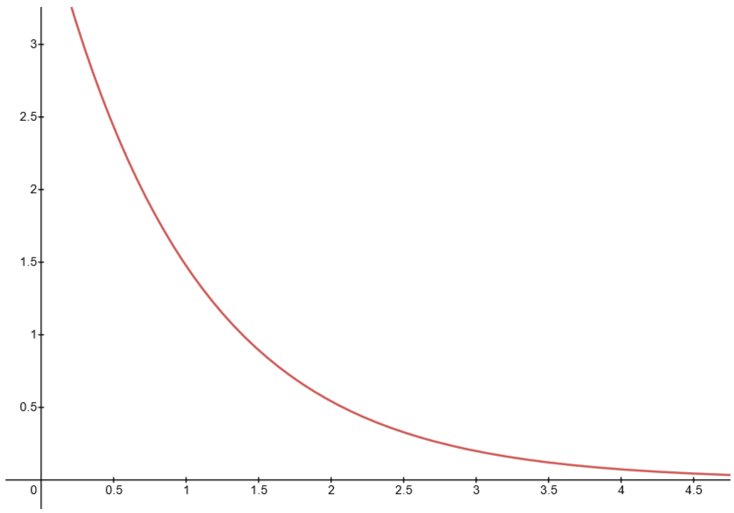
let $m = \frac{n}{i}$

$$\therefore = \left[\lim_{x\to\infty}(1+\frac{1}{m})^m\right] = e^{it}$$

Example: Money in the bank for 400 years

$$V(400) = 0.01e^{0.05 \cdot 400} = 5 \text{ million}$$

Example: Drug Metabolism



Decay will follow the following relation:

$$C = C_0 e^{-kt}$$

When on a course of drugs, you want the concentration to stay between toxic and therapeutic levels.

3 9.4 Models for Population Growth

$$\frac{dP}{dt} = kP \rightarrow P = P_0 e^{kt}$$

This model neglects certain factors, such as the fact that the environment can only support so many people/rabbits at a max.

The Logistic Model

$$\frac{dP}{dt} = kP(1 - \frac{P}{M})$$

M = carrying capacity of environment → max population.

The equation is considered separable, meaning:

$$\int \frac{dP}{P(1-\frac{P}{M})} = k\int dt$$

$$\frac{1}{P(1-\frac{P}{M})} = \frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$$

$$\int (\frac{1}{P} + \frac{1}{M-P})dP = k\int dt$$

$$ln\,|| \rightarrow ln\left|\frac{P}{M-P}\right| = kt + c$$

$$\frac{P}{M-P} = \pm e^{kt+c} \rightarrow \frac{M-P}{P} = Ae^{-kt}$$

$$A = \pm e^{-c}$$

Rearranging:

$$P(t) = \frac{M}{1+Ae^{-kt}}$$

First we get exponential growth, then an asymptotic to M.

4 9.5 Linear Equations

$$y' + p(x)y = q(x)$$

Linear 1^{st} order DE. Require that:
p(x), q(x) are continuous on some interval I. If q was zero, we would have a separable equation, so let's assume it's not.

Example:

$$xy' + y = x^2 \rightarrow y' + \frac{1}{x}y = x$$

LHS is just product rule of xy:

$$= (x \cdot y)'$$

Therefore easy to integrate, it's just xy

$$\rightarrow (xy)' = x^2$$

$$xy = \frac{x^3}{3} + c \to y = \frac{x^2}{3} + \frac{c}{x}$$

In general, try and make LHS (write in terms of) a simple derivative that we can then reverse it for the integral.

$$H(x) = \int p(x)dx$$

$$\frac{d}{dx}e^{h(x)} = e^{H(x)}\frac{d}{dx}H(x) = e^{H(x)}p(x)$$

$$\frac{d}{dx}(ye^{H(x)}) = y'e^{H(x)} + y^{H(x)}p(x)$$

$$= e^{H(x)}(y' + p(x)y)$$

LHS of DE = q(x).

$$\to \frac{d}{dx}(ye^{H(x)}) = e^{H(x)}q(x)$$

$$ye^{H(x)} = \int e^{Hx}q(x)dx + c$$

$$y = e^{-H(x)}\left[\int e^{H(x)}q(x)dx + c\right]$$

5 Steps to solving DE

1) Determine the correct formulation of the problem
2) Find the integrating factor

Example:

$$y' + 2y = 4$$

$$p(x) = 2$$

$$q(x) = 4$$

$$\to H(x) = \int 2dx = 2x \to \text{Integrating factor is: } IF = e^{2x}$$

We'll put the constant back in later. Now multiply through by integrating factor:

$$e^{2x}y' + 2e^{2x}y = e^{2x} \cdot 4 \to \frac{d}{dx}(e^{2x}y) = 4e^{2x}$$

$$e^{2x} \cdot y = \int 4e^{2x}dx + = \frac{4}{2}e^{2x} + c$$

or

$$y = 2 + ce^{-2x}$$

Example:

$$\frac{dT}{dt}\alpha T - \tau$$

T = temp of object
τ = surrounding temp
or

$$\frac{dT}{dt} = -k(T - \tau)$$

$$\therefore T' + kT = k\tau$$

$$p(t) = k$$

$$q(t) = k\tau$$

$$\to H(t) = \int (kdt = kt \to IF = ekt)$$

$$\frac{d}{dx}(e^{kt} \cdot T) = e^{kt} \cdot k\tau$$

$$\to T = ekT \left[\int e^{kt} \cdot kqdt + c \right]$$

Example:

$$xy' + 2y = 5x^3$$

$$y(1) = 0$$

Rearrange:

$$y' + \frac{2}{x}y = 5x^2$$

We now have:

$$p(x) = \frac{2}{x}$$

$$q(x) = 5x^2$$

$$H(x) = \int \frac{2}{x} dx = 2ln(x) = lnx^2 \to \text{IF } = e^{lnx^2} = x^2$$

$$x^2y' + 2xy = 5x^4 \to x^2y = 5\int x^4 dx + c = x^5 + c$$

$$\therefore y = x^3 + cx^{-2}$$

But we still need to satisfy our side-condition:

$$0 = 1 + c \to c = -1$$

$$\to y = x^3 - x^{-2}$$